XX. *A new Method of inveſtigating the Sums of Infinite Series.* *By the Rev.* Samuel Vince, *A. M. F. R. S.*

Read June 2, 1791.

THE ſummation of infinite ſeries is a ſubject, not only of curious ſpeculation, but alſo of the greateſt importance in the various branches of mathematics and philoſophy; in conſequence of which it has always claimed a very conſiderable ſhare of attention from the moſt celebrated mathematicians. I ſhall therefore make no apology for offering to the public the following new and very expeditious method, by which we may obtain the ſums of a great variety of ſeries, moſt of which have never before been treated of. As the ſummation depends on the ſums of the reciprocals of the powers of the natural numbers, tables of ſuch ſums are given as far as the 40th power to twelve places of decimals, by which the ſums of the ſeries will be found true to ten or eleven places; and if greater accuracy were required (which is a caſe that can very rarely happen) it might eaſily be obtained by continuing the tables. The firſt and third columns ſhew the ſums, and the ſecond and fourth the powers correſponding.

TABLE I.

Sum of $\frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} +$ &c. *ad infinitum.*

Sum	$n =$	Sum	$n =$
A = ,644934066848	2	W = ,000000238450	22
B = ,202056903159	3	X = ,000000119219	23
C = ,082323233711	4	Y = ,000000059608	24
D = ,036927755107	5	Z = ,000000029803	25
E = ,017343061984	6	A′ = ,000000014901	26
F = ,008349277387	7	B′ = ,000000007450	27
G = ,004077356198	8	C′ = ,000000003725	28
H = ,002008392826	9	D′ = ,000000001863	29
I = ,000994575128	10	E′ = ,000000000931	30
K = ,000494188604	11	F′ = ,000000000465	31
L = ,000246086553	12	G′ = ,000000000233	32
M = ,000122713347	13	H′ = ,000000000116	33
N = ,000061248135	14	I′ = ,000000000058	34
O = ,000030588236	15	K′ = ,000000000029	35
P = ,000015282259	16	L′ = ,000000000015	36
Q = ,000007637196	17	M′ = ,000000000007	37
R = ,000003817292	18	N′ = ,000000000004	38
S = ,000001908212	19	O′ = ,000000000002	39
T = ,000000953961	20	P′ = ,000000000001	40
V = ,000000476932	21		

TABLE II.

Sum of $\frac{1}{2^n} - \frac{1}{3^n} + \frac{1}{4^n} - \frac{1}{5^n} +$ &c. *ad infinitum.*

Sum	$n =$	Sum	$n =$
a = ,177532966576	2	w = ,000000238386	22
b = ,098457322630	3	x = ,000000119199	23
c = ,052967170503	4	y = ,000000059602	24
d = ,027880229587	5	z = ,000000029801	25
e = ,014448908703	6	a' = ,000000014901	26
f = ,007406180072	7	b' = ,000000007450	27
g = ,003766998147	8	c' = ,000000003725	28
h = ,001905702459	9	d' = ,000000001863	29
i = ,000960492403	10	e' = ,000000000931	30
k = ,000482856502	11	f' = ,000000000465	31
l = ,000242314856	12	g' = ,000000000233	32
m = ,000121457237	13	h' = ,000000000116	33
n = ,000060829654	14	i' = ,000000000058	34
o = ,000030448787	15	k' = ,000000000029	35
p = ,000015235790	16	l' = ,000000000015	36
q = ,000007621708	17	m' = ,000000000007	37
r = ,000003812130	18	n' = ,000000000004	38
s = ,000001906491	19	o' = ,000000000002	39
t = ,000000953389	20	p' = ,000000000001	40
v = ,000000476742	21		

TABLE III.

Sum of $\frac{1}{2^n}+\frac{1}{4^n}+\frac{1}{6^n}+$&c. *ad infinitum*.

Sum	$n=$	Sum	$n=$
A'' = ,411233516712	2	W'' = ,000000238419	22
B'' = ,150257112895	3	X'' = ,000000119209	23
C'' = ,067645202107	4	Y'' = ,000000059605	24
D'' = ,032403992347	5	Z'' = ,000000029802	25
E'' = ,015895985344	6	A''' = ,000000014901	26
F'' = ,007877728730	7	B''' = ,000000007450	27
G'' = ,003922177173	8	C''' = ,000000003725	28
H'' = ,001957047643	9	D''' = ,000000001863	29
I'' = ,000977533765	10	E''' = ,000000000931	30
K'' = ,000488522553	11	F''' = ,000000000465	31
L'' = ,000244200705	12	G''' = ,000000000233	32
M'' = ,000122085292	13	H''' = ,000000000116	33
N'' = ,000061038895	14	I''' = ,000000000058	34
O'' = ,000030518512	15	K''' = ,000000000029	35
P'' = ,000015259024	16	L''' = ,000000000015	36
Q'' = ,000007629452	17	M''' = ,000000000007	37
R'' = ,000003814712	18	N''' = ,000000000004	38
S'' = ,000001907352	19	O''' = ,000000000002	39
T'' = ,000000953675	20	P''' = ,000000000001	40
V'' = ,000000476837	21		

TABLE

TABLE IV.

Sum of $\frac{1}{3^n}+\frac{1}{5^n}+\frac{1}{7^n}+$ &c. *ad infinitum.*

Sum	$n=$	Sum	$n=$
a'' = ,233700550136	2	n'' = ,000000209240	14
b'' = ,051799790264	3	o'' = ,000000069724	15
c'' = ,014678031604	4	p'' = ,000000023234	16
d'' = ,004523762760	5	q'' = ,000000007744	17
e'' = ,001447076640	6	r'' = ,000000002581	18
f'' = ,000471548657	7	s'' = ,000000000864	19
g'' = ,000155179025	8	t'' = ,000000000286	20
h'' = ,000051345183	9	v'' = ,000000000095	21
i'' = ,000017041362	10	w'' = ,000000000032	22
k'' = ,000005666051	11	x'' = ,000000000011	23
l'' = ,000001885848	12	y'' = ,000000000004	24
m'' = ,000000628055	13	z'' = ,000000000001	25

PROP. I.

To find the ſum of the ſums of the reciprocal ſquares, cubes, &c. &c. ad infinitum.

By diviſion $\frac{1}{x-1\times x}=\frac{1}{x^2}+\frac{1}{x^3}+\frac{1}{x^4}+$ &c. *ad inf.*; hence if we make each of theſe terms the general term of a ſeries, and write 2, 3, 4, &c. *ad inf.* for x, we have $\frac{1}{1\,.\,2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+$ &c. = (Table 1.) A + B + C + D + &c.; but $\frac{1}{1\,.\,2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+$ &c. *ad inf.* = 1; hence A + B + C + D + &c. *ad inf.* = 1.

As $\frac{1}{x \times \overline{x+1}} = \frac{1}{x^2} - \frac{1}{x^3} + \frac{1}{x^4} - \frac{1}{x^5} + \&c.$ *ad inf.*; we have, by the fame method of proceeding, $A - B + C - D + \&c.$ *ad inf.* $= \frac{1}{2}$; confequently $A + C + E + \&c. = \frac{3}{4}$, and $B + D + F + \&c. = \frac{1}{4}$.

Becaufe $\frac{1}{\overline{x-1} \times x} = \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4} + \&c.$ *ad inf.*; if for x we write 2, 4, 6, &c., then will $\frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} + \&c. =$ (Tab. 3) $A'' + B'' + C'' + D'' + \&c.$; but $\frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} + \&c. =$ hyp. log. 2; hence $A'' + B'' + C'' + D'' + \&c. =$ hyp. log. 2.

If in the fame expreffion we write 3, 5, 7, &c. for x, then $\frac{1}{2 \cdot 3} + \frac{1}{4 \cdot 5} + \frac{1}{6 \cdot 7} + \&c. =$ (Tab. 4.) $a'' + b'' + c'' + \&c.$; but $\frac{1}{2 \cdot 3} + \frac{1}{4 \cdot 5} + \frac{1}{6 \cdot 7} + \&c. = 1 -$ hyp. log. 2; hence $a'' + b'' + c'' + \&c. = 1 -$ hyp. log. 2.—Hence from either of thefe two laft cafes, we have a very expeditious method of finding the hyp. log. 2.

PROP. II.

To find the fum of the infinite feries whofe general term is $\frac{1}{mx^r \pm n}$.

By divifion $\frac{1}{mx^r \pm n} = \frac{1}{mx^r} \mp \frac{n}{m^2 x^{2r}} + \frac{n^2}{m^3 x^{3r}} \mp \frac{n^3}{m^4 x^{4r}} + \&c.$ *ad inf.*; hence, if $\frac{1}{mx^r \pm n}$ be made the general term of a feries, and for x we write 2, 3, 4, &c., its fum will be equal to the fums of another fet of feriefes, whofe terms are the powers of the reciprocals of the natural numbers refpectively multiplied into

into $\frac{1}{m}$, $\frac{n}{m^2}$, $\frac{n^2}{m^3}$, &c.; hence the ſum of each of theſe ſeries being known from the tables, the ſum of the given ſeries will be found.

Ex. 1. Let $\frac{1}{x^2+1}$ be the general term; now $\frac{1}{x^2+1} = \frac{1}{x^2} - \frac{1}{x^4} + \frac{1}{x^6} - \frac{1}{x^8} +$ &c.; hence if for x we write 2, 3, 4, &c. we have $\frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \frac{1}{26} +$ &c. $= A - C + E - G +$ &c. $=$ (by Tab. 1.) ,5766740374 69.

Ex. 2. Let $\frac{1}{x^2-1}$ be the general term; then, by the ſame method of proceeding, $\frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} +$ &c. $= A + C + E +$ &c. $=$ (by Prop. 1.) $\frac{3}{4}$.

Cor. Becauſe $\frac{1}{8} + \frac{1}{24} + \frac{1}{48} +$ &c. $= \frac{1}{8} \times \overline{1 + \frac{1}{3} + \frac{1}{6} + \text{\&c.}} =$ (as $1 + \frac{1}{3} + \frac{1}{6} +$ &c. is the reciprocal of the figurative numbers of the ſecond order) $\frac{1}{8} \times 2 = \frac{1}{4}$; therefore $\frac{1}{3} + \frac{1}{15} + \frac{1}{35} +$ &c. $= \frac{1}{2}$. Alſo, as $\frac{1}{x^2-1} = \frac{1}{x^2} + \frac{1}{x^4} + \frac{1}{x^6} +$ &c.; if we write 2, 4, 6, &c. for x, we have $\frac{1}{3} + \frac{1}{15} + \frac{1}{35} +$ &c. $=$ (by Tab. 3.) $A'' + C'' + E'' +$ &c. $= \frac{1}{2}$; but, by Prop. 1. $A'' + B'' + C'' + D'' +$ &c. $=$ hyp. log. 2; hence $B'' + D'' + F'' +$ &c. $= -\frac{1}{2} +$ hyp. log. 2.

Ex. 3. Let the general term be $\frac{1}{x^3-1} = \frac{1}{x^3} + \frac{1}{x^6} + \frac{1}{x^9} +$ &c., and, by writing 2, 3, 4, &c. for x, we have $\frac{1}{7} + \frac{1}{26} + \frac{1}{63} +$ &c. $= B + E + H +$ &c. $=$,221689395104.

Ex.

Ex. 4. Let the general term be $\frac{1}{3x^4-2}=\frac{1}{3x^4}+\frac{2}{9x^8}+\frac{4}{27x^{12}}+$ &c., and, by writing 2, 3, 4, &c. for x, &c. we have $\frac{1}{46}+\frac{1}{241}+\frac{1}{766}+$ &c. $\frac{1}{3}C+\frac{2}{9}G+\frac{4}{27}L+$ &c. $=,0283852252052$.

Ex. 5. To find the sum of the series $\frac{1}{9}-\frac{1}{26}+\frac{1}{65}-\frac{1}{124}+$ &c. If we write 2, -3, 4, -5, &c. for x, the general term will be $\frac{1}{x^3+1}=\frac{1}{x^3}-\frac{1}{x^6}+\frac{1}{x^9}-\frac{1}{x^{12}}+$ &c. Now, by writing 2, -3, 4, -5, &c. for x, the serieses of which $\frac{1}{x^3}$, $\frac{1}{x^9}$, &c. are the general terms, will be alternately + and $-$, and therefore their sums will be found in Tab. 2. and the serieses of which $\frac{1}{x^6}$, $\frac{1}{x^{12}}$, &c. are the general terms will have their terms all +, and therefore their sums will be found in Tab. 1. Hence the sum required $= b+h+o+$ &c. $-E-L-R-$ &c. $=$,0828009318O3.

PROP. III.

To find the sum of the sums of the reciprocals of the odd *powers in* Tab. 2.

By division $\frac{1}{\overline{x-1}\times x}=\frac{1}{x^2}+\frac{1}{x^3}+\frac{1}{x^4}+\frac{1}{x^5}+$ &c.; hence by writing 2, -3, 4, -5, &c. for x, the sums of the serieses of which $\frac{1}{x^3}$, $\frac{1}{x^5}$, &c. are the general terms, may be found by Tab. 2. and the other sums by Tab. 1.; hence $\frac{1}{1\cdot 2}+\frac{1}{3\cdot 4}+\frac{1}{5\cdot 6}+$ &c. $=A+C+E+$ &c. $+b+d+f+$ &c.; but $\frac{1}{1\cdot 2}+\frac{1}{3\cdot 4}+\frac{1}{5\cdot 6}+$ &c.

&c $= -\frac{1}{2} + 2$ hyp. log. 2; and by Prop. 1. $A + C + E +$ &c. $= \frac{3}{4}$; hence $b + d + f +$ &c. $= -\frac{5}{4} + 2$ hyp. log. 2.

PROP. IV.

To find the sum of the infinite series whose general term is $\frac{x^s}{mx^r \pm n}$.

By division $\frac{x^s}{mx^r \pm n} = \frac{1}{mx^{r-s}} \mp \frac{n}{m^2 x^{2r-s}} + \frac{n^2}{m^3 x^{3r-s}} \mp$ &c. *ad inf.*; hence the sum of the series of which $\frac{x^s}{mx^r \pm n}$ is the general term, is found as in Prop. 2. Here r must be greater than s at least by 2, otherwise the sum will be infinite.

Ex. 1. Let the general term be $\frac{x^2}{x^4+1} = \frac{1}{x^2} - \frac{1}{x^6} + \frac{1}{x^{10}} -$ &c.; hence if for x we write 2, 3, 4, &c. we have $\frac{4}{17} + \frac{9}{82} + \frac{16}{257} +$ &c. $= A - E + I - N +$ &c. $= ,538527924723$.—If for x we write 2, 4, 6, &c. we get $\frac{4}{17} + \frac{16}{257} + \frac{36}{1296} +$ &c. $= A'' - E'' + I'' - N'' +$ &c. $= ,396257616555$.

Ex. 2. Let the general term be $\frac{x}{3x^3-1} = \frac{1}{3x^2} + \frac{1}{9x^5} + \frac{1}{27x^8} +$ &c.; hence if we write 2, 3, 4, &c. for x, we have $\frac{2}{23} + \frac{3}{80} + \frac{4}{191} +$ &c. $= \frac{1}{3}A + \frac{1}{9}D + \frac{1}{27}G +$ &c. $= ,219238483448$.

By this proposition we may find the sum of any series whose general term is $\frac{ax^s + bx^{s-1} + cx^{s-2} + \text{\&c.}}{mx^r \pm n}$; for this resolves itself into $\frac{ax^s}{mx^r \pm n}$, $\frac{bx^{s-1}}{mx^r \pm n}$, &c. &c., the sum of each of which series is

found by this propofition. Now the $\overline{s+1}$th differences of the numerators of this general term are $=0$, and therefore it comprehends all feries under fuch circumftances. For example, let the given feries be $\frac{4}{17}+\frac{13}{82}+\frac{26}{257}+\frac{43}{626}$. Here the third differences of the numerators $=0$; to find therefore the general expreffion for the numerator, affume ax^2+bx+c for it; and, by writing 2, 3, 4, for x, we have $4a+2b+c=4$, $9a+3b+-c=13$, $16a+4b+c=26$; hence $a=2$, $b=-1$, $c=-2$; and as the denominator is manifeftly x^4+1, the general term will be $\frac{2x^2-x-2}{x^4+1}=\frac{2x^2}{x^4+1}-\frac{x}{x^4+1}-\frac{2}{x^4+1}$, each of which being made the general term of a feries, their fum will be found to be refpectively 1,077055849446, 0,194173022145, and 0,156955159332; hence the fum of the given feries is 0,725927667969.

If s be negative, the general term becomes $\frac{1}{x^s \times \overline{mx^r \pm n}}=\frac{1}{mx^{r+s}}$ $\mp\frac{1}{m^2x^{2r+s}}+\frac{1}{m^3x^{3r+s}}\mp$ &c.

Ex. 1. To find the fum of $\frac{1}{1.2.3}-\frac{1}{2.3.4}+\frac{1}{3.4.5}-$ &c. *ad inf.* Here the general term is $\frac{1}{\overline{x-1}\times x\times\overline{x+1}}=\frac{1}{x\times\overline{x^2-1}}=$ $\frac{1}{x^3}+\frac{1}{x^5}+\frac{1}{x^7}+$&c.; hence, by writing $2, -3, 4, -5$, &c. for x, we have the fum $=b+d+f+$ &c. $=$ (by Prop. 3.) $-\frac{5}{4}+2$ hyp. log. 2.

If $\frac{1}{\overline{x-1}\times x^3\times\overline{x+1}}$ be the general term it refolves itfelf into $\frac{1}{x^5}+\frac{1}{x^7}+\frac{1}{x^9}+$&c.; confequently the fum of $\frac{1}{1.2^3.3}-\frac{1}{2.3^3.4}$

+

$+\frac{1}{3\cdot 4^3\cdot 5}-$ &c. $=-b-\frac{5}{4}+2$ hyp. log. 2. In like manner the sum of $\frac{1}{1\cdot 2^5\cdot 3}-\frac{1}{2\cdot 3^5\cdot 4}+\frac{1}{3\cdot 4^5\cdot 5}-$ &c. $=-b-d-\frac{5}{4}+$ 2 hyp. log. 2. Thus we may proceed as far as we please by adding two powers to the middle term; and hence this remarkable property of the serieses, that the difference of the sums of the serieses where the middle term is x, x^3, x^5, &c. is b, d, f, &c. respectively.

Ex. 2. In like manner if the general term be $\frac{1}{\overline{x-1}\times x^3\times\overline{x+1}}$, and we write 2, 3, 4, &c. for x, we have $\frac{1}{1\cdot 2^3\cdot 3}+\frac{1}{2\cdot 3^3\cdot 3}+\frac{1}{3\cdot 4^3\cdot 5}+$ &c. $=D+F+H+$&c. $=$ (by Prop. 1.) $\frac{1}{4}-B$. Hence also $\frac{1}{1\cdot 2^5\cdot 3}+\frac{1}{2\cdot 3^5\cdot 4}+$ &c. $=\frac{1}{4}-B-D$; and so on as before.

If the general term be under the form $\frac{1}{x^n\cdot\overline{x+m}}$, it will be most convenient to resolve it thus: by division $\frac{1}{x+m}=\frac{1}{x}-\frac{m}{x^2}+\frac{m^2}{x^3}-$ &c. $\pm\frac{m^n}{x^n\cdot\overline{x+m}}$; hence $\pm\frac{1}{x^n\cdot\overline{x+m}}=\overline{\frac{1}{x+m}-\frac{1}{x}+\frac{m}{x^2}-\frac{m^2}{x^3}+\text{\&c.}}\times\frac{1}{m^n}=\overline{-\frac{m}{x\cdot\overline{x+m}}+\frac{m}{x^2}-\frac{m^2}{x^3}+\text{\&c.}}\times\frac{1}{m^n}$, where the sign on the left hand will be $+$ or $-$ according as n is even or odd, and the number of terms on the right is $=n$. Now the sum of the series whose general term is $\frac{m}{x\cdot\overline{x+m}}$ is well known, and the sums of the other are found from the tables.

Ex. 1. To find the sum of $\frac{1}{2^2\cdot 3}+\frac{1}{3^2\cdot 4}+\frac{1}{4^2\cdot 5}+$ &c. *ad inf.*

 Here

Here the general term is $\frac{1}{x^2 \times \overline{x+1}} = -\frac{1}{x \cdot \overline{x+1}} + \frac{1}{x^2}$, and by writing 2, 3. 4, &c. for x, we have the sum $= -\frac{1}{2 \cdot 3} - \frac{1}{3 \cdot 4} - \&c. + A = -\frac{1}{2} + A$. In like manner $\frac{1}{2^3 \cdot 3} + \frac{1}{4^2 \cdot 5} + \frac{1}{6^2 \cdot 7} + \&c. = -1 + \overline{\text{hyp.log.} 2 + A''}$. Also $\frac{1}{2^3 \cdot 5} + \frac{1}{3^3 \cdot 6} + \frac{1}{4^3 \cdot 7} + \&c. = \overline{\frac{13}{12} - 3A + 9B} \times \frac{1}{27}$.

If m be negative, then $\frac{1}{x^n \cdot \overline{x-m}} = \overline{\frac{m}{x \cdot \overline{x-m}} - \frac{m}{x^2} - \frac{m^2}{x^3} - \&c.} \times \frac{1}{m^n}$. Hence $\frac{1}{2^4 \cdot 1} + \frac{1}{3^4 \cdot 2} + \frac{1}{4^4 \cdot 3} + \&c. = 1 - A - B - C$; and so on for others of the same kind.

If the general term be under this form $\frac{1}{x^{rn} \cdot \overline{ax^n + m}}$, then, in like manner, we have $\pm \frac{1}{x^{rn} \cdot \overline{ax^n + m}} = \overline{\frac{1}{ax^n + m} - \frac{1}{ax^n} + \frac{m}{a^2 x^{2n}} - \&c.} \times \frac{a^r}{m^r}$, where the sign on the left hand will be + or −, according as r is even or odd, and the number of terms on the right is $= r + 1$.

Ex. 1. To find the sum of $\frac{1}{2^4 \cdot 5} + \frac{1}{3^4 \cdot 10} + \frac{1}{4^4 \cdot 17} + \&c.$ Here $m = 1$, $n = 2$, $r = 2$, $a = 1$, and the general term $\frac{1}{x^4 \times \overline{x^2 + 1}} = \frac{1}{x^2 + 1} - \frac{1}{x^2} + \frac{1}{x^4}$; now the sum of the series whose general term is $\frac{1}{x^2 + 1}$ is $= ,576674037469$, by Prop. 2.; consequently the sum required $= ,576674037469 - A + C = ,014063204332$.

Ex. 2. If the given series be $\frac{1}{4 \cdot 5} + \frac{1}{9 \cdot 10} + \frac{1}{16 \cdot 17} + \&c.$ the general

general term will be $\frac{1}{x^2 \cdot \overline{x+1}} = -\frac{1}{x^2+1} + \frac{1}{x^2}$; hence, by writing 2, 3, 4, &c. for x, we have the sum $= -,576674037469 + A = ,0682600293 8$.

If m be negative, then $\frac{1}{x^{rn} \cdot \overline{ax^n - m}} = \overline{\frac{1}{ax^n - m} - \frac{1}{ax^n} - \frac{m}{a^2 x^{2n}} - \&c.}$ $\times \frac{a^r}{m^r}$.

Ex. 1. To find the sum of $\frac{1}{1 \cdot 2^2 \cdot 3} + \frac{1}{2 \cdot 3^2 \cdot 4} + \frac{1}{3 \cdot 4^2 \cdot 5} + \&c.$ Here the general term is $\frac{1}{\overline{x-1} \times x^2 \times \overline{x+1}} = \frac{1}{x^2 \times \overline{x^2 - 1}} = \frac{1}{x^2 - 1} - \frac{1}{x^2}$; now, by writing 2, 3, 4, &c. for x, the sum of the series whose general term is $\frac{1}{x^2 - 1}$ is $= \frac{3}{4}$, by Prop. 2.; hence the sum required $= \frac{3}{4} - A$.

Ex. 2. Let the given series be $\frac{1}{1 \cdot 2^2 \cdot 3} + \frac{1}{3 \cdot 4^2 \cdot 5} + \frac{1}{5 \cdot 6^2 \cdot 7} +$ &c. Here the general term is the same as before, writing 2, 4, 6, &c. for x; and, by Prop. 2. the sum of the series whose general term is $\frac{1}{x^2 - 1}$ is $= \frac{1}{2}$; hence the sum $= \frac{1}{2} - A''$.

Ex. 3. In like manner the sum of the series $\frac{1}{1 \cdot 2^4 \cdot 3} + \frac{1}{2 \cdot 3^4 \cdot 4} + \frac{1}{3 \cdot 4^4 \cdot 5} + \&c. = ,221689395104 - B$.

Ex. 4. To find the sum of $\frac{1}{3 \cdot 4^2 \cdot 5} + \frac{1}{8 \cdot 9^2 \cdot 10} + \frac{1}{15 \cdot 16^2 \cdot 17} + \&c.$ Here the general term is $\frac{1}{\overline{x^2 - 1} \times x^4 \times \overline{x^2 + 1}} = \frac{1}{x^4 \times \overline{x^4 - 1}} = \frac{1}{x^4 - 1} - \frac{1}{x^4}$; but the sum of the series whose general term is $\frac{1}{x^4 - 1}$ is $= ,086662976264$; hence the sum required $= ,086662976264 - C$.

PROP. V.

To find the ſum of the infinite ſeries $\frac{1}{15}+\frac{1}{40}+\frac{1}{85}+\frac{1}{156}+\frac{1}{259}+$ &c.

In this ſeries the fourth differences of the denominators $=0$; therefore the general term muſt be repreſented by $\frac{1}{ax^3+bx^2+cx+d}$; write therefore 2, 3, 4, &c. for x, and we have $8a+4b+2c+d=15$, $27a+9b+3c+d=40$, $64a+16b+4c+d=85$, $125a+25b+5c+d=156$; hence $a=1$, $b=1$, $c=1$, $d=1$, and the general term is $\frac{1}{x^3+x^2+x+1}=\frac{1}{x^3}-\frac{1}{x^4}+\frac{1}{x^7}-\frac{1}{x^8}+$ &c.; hence the ſum $=B-C+F-G+K-L+$ &c. $=,1242700165$.

PROP. VI.

To find the ſum of $\frac{2}{5^2}+\frac{3}{10^2}+\frac{4}{17^2}+$ &c. *ad inf.*

The general term $=\frac{x}{\overline{x^2+1}^2}=\frac{1}{x^3}-\frac{2}{x^5}-\frac{3}{x^7}+$ &c.; hence, by writing 2, 3, 4, &c. for x, we have the ſum $=B-2D+3F-$ &c. $=,147115771469$.

In like manner $\frac{2}{3^2}+\frac{3}{8^2}+\frac{4}{15^2}+$ &c. $=B+2D+3F+$ &c. $=,312498999865$.

PROP.

PROP. VII.

To find the sum of $\frac{1}{3^2 \cdot 5^2} + \frac{1}{8^2 \cdot 10^2} + \frac{1}{15^2 \cdot 17^2} +$ &c. *ad inf.*

The general term is $\frac{1}{\overline{x^2-1}^2 \times \overline{x^2+1}^2} = \frac{1}{x^8} + \frac{2}{x^{12}} + \frac{3}{x^{16}} +$ &c.; hence, by writing 2, 3, 4, &c. for x, we have the sum $=$ G + 2L + 3P + &c. $=$,009447690684.

PROP. VIII.

To find the sum of $\frac{1}{1^3 \cdot 2^3 \cdot 3^3} + \frac{1}{2^3 \cdot 3^3 \cdot 4^3} + \frac{1}{3^3 \cdot 4^3 \cdot 5^3} +$ &c. *ad inf.*

Here the general term is $\frac{1}{\overline{x-1}^3 \cdot x^3 \cdot \overline{x+1}^3} = \frac{1}{x^9} + \frac{3}{x^{11}} + \frac{6}{x^{13}} +$ &c. and hence the sum $=$ H + 3K + 6M + &c. $=$,004707148337.

PROP. IX.

To find the sum of the infinite series $1 - \frac{1}{3} + \frac{1}{6} - \frac{1}{10} +$ &c. *being a series of the reciprocal of the figurative numbers of the* 3rd *order, having the signs alternately* + *and* −.

This series, by resolving two terms into one, becomes $\frac{4}{1 \cdot 2 \cdot 3} + \frac{4}{3 \cdot 4 \cdot 5} + \frac{4}{5 \cdot 6 \cdot 7} +$ &c. whose general term, by writing 2, 4, 6, &c. for x, is $\frac{4}{x-1 \times x \times x+1} = \frac{4}{x^3} + \frac{4}{x^5} + \frac{4}{x^7} +$ &c. consequently the sum $= 4B'' + 4D'' + 4F'' +$ &c. $=$ (by Cor. Ex. 2. Prop. 2.) $-2 + 4$ hyp. log. 2.

Cor.

Cor. Hence, as $1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\&c.=2$, we have $1+\frac{1}{6}+\frac{1}{15}+\&c.=2$ hyp. log. 2, and $\frac{1}{3}+\frac{1}{10}+\frac{1}{21}+\&c.=2-2$ hyp. log. 2.

PROP. X.

To find the ſum of the infinite ſeries $1-\frac{1}{4}+\frac{1}{10}-\frac{1}{20}+\&c.$ *being the reciprocals of the figurative numbers of the 4th order, having the ſigns alternately* $+$ *and* $-$.

If we write $2, -3, 4, -5$, &c. for x, the general term will be $\frac{6}{x^3-x}=\frac{6}{x^3}+\frac{6}{x^5}+\frac{6}{x^7}+\&c.$; hence the ſum required $=6b+6d+6f+\&c.=$ (by Prop. 3.) $-7\frac{1}{2}+12$ hyp. log. 2.

Cor. Becauſe the ſum of $1+\frac{1}{4}+\frac{1}{10}+\frac{1}{20}+\&c.=\frac{3}{2}$; therefore $1+\frac{1}{10}+\frac{1}{35}+\&c.=-3+6$ hyp. log. 2; and $\frac{1}{4}+\frac{1}{20}+\frac{1}{56}+\&c.=4\frac{1}{2}-6$ hyp. log. 2.

PROP. XI.

To find the ſum of $\frac{2^2}{1^2\cdot 3^2}+\frac{3^2}{2^2\cdot 4^2}+\frac{4^2}{3^2\cdot 5^2}+\&c.$ *ad infinitum.*

The general term, by writing 2, 3, 4, &c. for x, is $\frac{x^2}{\overline{x-1}^2\times\overline{x+1}^2}=\frac{1}{x^2}+\frac{2}{x^4}+\frac{3}{x^6}+\&c.$; hence the ſum $=A+2C+3E+\&c.=,884966993407$.

PROP. XII.

To find the sum of $\frac{1}{1 \cdot 2^2 \cdot 3^3}+\frac{1}{2 \cdot 3^2 \cdot 4^3}+\frac{1}{3 \cdot 4^2 \cdot 5^3}+$ &c. *ad infinitum.*

Here the general term, by writing 2, 3, 4, &c. for x, is $\frac{1}{\overline{x-1} \cdot x^2 \cdot \overline{x+1}^3}=\frac{1}{x^6}-\frac{2}{x^7}+\frac{4}{x^8}+\frac{6}{x^9}-\frac{9}{x^{10}}-\frac{12}{x^{11}}+\frac{16}{x^{12}}-$ &c.; consequently the sum $= E-2F+4G-6H+9I-12K+$ &c. $=$,0103708984 82.

PROP. XIII.

To find the sum of $\frac{1}{2}A-\frac{1}{4}B+\frac{1}{8}C-$&c. *ad infinitum.*

The hyp. log. $2=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-$ &c. $=1+\frac{1}{3}+\frac{1}{5}+$ &c. $-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}-$&c.; hence $2 \times$ hyp. log. 2, or hyp. log. 4, $=\frac{2}{1}+\frac{2}{3}+\frac{2}{5}+$&c. $-1-\frac{1}{2}-\frac{1}{3}-$ &c. Now, by division, $\frac{2}{2x+1}=\frac{1}{x}-\frac{1}{2x^2}+\frac{1}{4x^3}-\frac{1}{8x^4}+$&c.; hence, by writing 2, 3, 4, &c. for x, we have (after transposition) $\frac{2}{5}+\frac{2}{7}+$&c. $-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-$&c. $=-\frac{1}{2}A+\frac{1}{4}B-\frac{1}{8}C+$&c.; hence, by adding equal quantities to each side, we have $\frac{2}{1}+\frac{2}{3}+\frac{2}{5}+$&c. $-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-$ &c. $=\frac{8}{3}-\frac{1}{2}A+\frac{1}{4}B-\frac{1}{8}C+$&c., consequently $\frac{1}{2}A-\frac{1}{4}B+\frac{1}{8}C-$ &c. $=\frac{8}{3}-\frac{2}{1}-\frac{2}{3}-\frac{2}{5}-$&c. $+1+\frac{1}{2}+\frac{1}{3}+$&c. $=\frac{8}{3}-$hyp. log. 4.

PROP. XIV.

To find the ſum of the infinite ſeries $\frac{1}{2\cdot 5}+\frac{1}{3\cdot 7}+\frac{1}{4\cdot 9}+$ &c.

The general term, by writing 2, 3, 4, &c. for x, is $\frac{1}{x\,.\,\overline{2x+1}}=\frac{1}{2x^2}-\frac{1}{4x^3}+\frac{1}{8x^4}-$ &c.; hence the ſum $=\frac{1}{2}A-\frac{1}{4}B+\frac{1}{8}C$ $-$ &c. $=$ (by Prop. 13.) $\frac{8}{3}-$ hyp. log. 4.

PROP. XV.

To find the ſum of $1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots$ *to* $\frac{1}{x}$.

The hyp. log. $\frac{x}{x-1}=\frac{1}{x}+\frac{1}{2x^2}+\frac{1}{3x^3}+\frac{1}{4x^4}+$ &c.; conſequently hyp. log. $\frac{x}{x-1}-\frac{1}{2x^2}-\frac{1}{3x^3}-\frac{1}{4x^4}-$ &c. $=\frac{1}{x}$; hence, if we write 2, 3, 4, &c. for x, we have hyp. log. $\frac{2}{1}+$ hyp. log. $\frac{3}{2}+$ &c. . . . .

$$\left.\begin{array}{l}\text{hyp. log. }\frac{x}{x-1}-\frac{1}{2}\times\overline{\frac{1}{2^2}+\frac{1}{3^2}+\&c.\ldots\ldots\frac{1}{x^2}}\\ \qquad\qquad -\frac{1}{3}\times\overline{\frac{1}{2^3}+\frac{1}{3^3}+\&c.\ldots\ldots\frac{1}{x^3}}\\ \qquad\qquad -\frac{1}{4}\times\overline{\frac{1}{2^4}+\frac{1}{3^4}+\&c.\ldots\ldots\frac{1}{x^4}}\\ \qquad\qquad -\ \&c.\quad \&c.\qquad \&c.\end{array}\right\}=$$

$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+$ &c. . . . $\frac{1}{x}$; but hyp. log. $\frac{2}{1}+$ hyp. log. $\frac{3}{2}+$ hyp. log. $\frac{4}{3}+$ &c. . . . . hyp. log. $\frac{x}{x-1}=$ hyp. log. $\frac{2}{1}\times\frac{3}{2}\times\frac{4}{3}\times$ &c. . . . . $\frac{x}{x-1}=$ hyp. log. x; alſo $\frac{1}{2^2}+\frac{1}{3^2}+$ &c. . . . . $\frac{1}{x^2}=$ the

 ſum

sum of the same series *ad infinitum*, *minus* the sum of all the terms from $\frac{1}{x^2}$ = (if $x+1=n$) $A - \frac{1}{n} - \frac{1}{2n^2} - \frac{1}{6n^3} + \frac{1}{30n^5} - \frac{1}{42n^7} +$ &c.; in the same manner $\frac{1}{2^3} + \frac{1}{3^3} +$ &c. . . . . $\frac{1}{x^3} = B - \frac{1}{2n^2} - \frac{1}{2n^3} - \frac{1}{4n^4} + \frac{1}{12n^6} - \frac{1}{12n^8} +$ &c.; and so on for the other serieses; hence, by substitution, and adding unity to each side, we have hyp. log. $x + 1 - \frac{1}{2}A - \frac{1}{3}B - \frac{1}{4}C -$ &c. $+ \frac{1}{2n} + \frac{5}{12n^2} + \frac{1}{3n^3} + \frac{31}{120n^4} + \frac{1}{5n^5} + \frac{41}{252n^6} + \frac{1}{7n^7} + \frac{31}{240n^8} +$ &c. $= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} +$ &c. . . . $\frac{1}{x}$; but $1 - \frac{1}{2}A - \frac{1}{3}B - \frac{1}{4}C -$ &c. $= ,577215664901$; hence $1 + \frac{1}{2} + \frac{1}{3} +$ &c. . . . . $\frac{1}{x}$ = hyp. log. $x + ,577215664901 + \frac{1}{2n} + \frac{5}{12n^2} + \frac{1}{3n^3} + \frac{31}{120n^4} + \frac{1}{5n^5} + \frac{41}{252n^6} + \frac{1}{7n^7} + \frac{31}{240n^8} +$ &c.

Ex. 1. Let $x = 10000$; then

$$\begin{array}{rl} \text{hyp. log. } 10000 = & 9,210340371976 \\ \text{const. quant.} = & ,577215664901 \\ \frac{1}{2n} = & ,000049995000 \\ \frac{5}{12n^2} = & ,000000004166 \\ \hline \text{therefore the sum required} = & 9,787606036043 \\ \hline \end{array}$$

Ex. 2. Let $x = 10000000$; then

$$\begin{array}{rl} \text{hyp. log. } 10000000 = & 16,118095650958 \\ \text{const. quant.} = & ,577215664901 \\ \frac{1}{2n} = & ,000000049999 \\ \hline \text{therefore the sum required} = & 16,695311365858 \\ \hline \end{array}$$

PROP. XVI.

To find the value of $\alpha \times \beta \times \gamma \times \delta \times$ &c. *ad infinitum, supposing the general term to be a rational function of* x.

Let π be the general term, then resolve $\frac{\dot{\pi}}{\pi}$ into an infinite series, and take the fluent on both sides; then write 2, 3, 4, &c. for x, and one side will become the hyp. log. of the given series, and the value of the other side may be found from the tables.

Ex. 1. To find the value of $\frac{4}{3} \times \frac{9}{8} \times \frac{16}{15} \times$ &c. *ad infinitum.*

Here the general term is $\frac{x^2}{x^2-1}$; hence $\frac{\dot{\pi}}{\pi} = -\frac{2\dot{x}}{x^3-x} = -\frac{2\dot{x}}{x^3} - \frac{2\dot{x}}{x^5} - \frac{2\dot{x}}{x^7} -$ &c.; hence the hyp. log. $\frac{x^2}{x^2-1} = \frac{1}{x^2} + \frac{1}{2x^4} + \frac{1}{3x^6} +$ &c. Write 2, 3, 4, &c. for x, and we have the hyp. log. $\frac{4}{3}$ + hyp. log. $\frac{9}{8}$ + hyp. log. $\frac{16}{15}$ + &c. $= A + \frac{1}{2}C + \frac{1}{3}E +$ &c. $= ,693147180574$, which is the hyp. log. 2; but hyp. log. $\frac{4}{3}$ + hyp. log. $\frac{9}{8}$ + hyp. log. $\frac{16}{15}$ + &c. = hyp. log. $\frac{4}{3} \times \frac{9}{8} \times \frac{16}{15} \times$ &c. consequently $\frac{4}{3} \times \frac{9}{8} \times \frac{16}{15} \times$ &c. $= 2$.

Ex. 2. To find the value of $\frac{8}{7} \times \frac{27}{26} \times \frac{64}{63} \times$ &c. *ad infinitum.*

Here the general term is $\frac{x^3}{x^3-1}$; hence $\frac{\dot{\pi}}{\pi} = -\frac{3\dot{x}}{x^4-x} = -\frac{3\dot{x}}{x^4} - \frac{3\dot{x}}{x^7} - \frac{3\dot{x}}{x^{10}} -$ &c.; hence the hyp. log. $\frac{x^3}{x^3-1} = \frac{1}{x^3} + \frac{1}{2x^6} + \frac{1}{3x^9} +$ &c. Write 2, 3, 4, &c. for x, and we have hyp. log. $\frac{8}{7}$ + hyp. log.

$\frac{27}{26}$

$\frac{27}{26}$ + hyp. log. $\frac{64}{63}$ + &c. = B + $\frac{1}{2}$E + $\frac{1}{3}$H + &c. = ,211466250444; or hyp. log. $\frac{8}{7} \times \frac{27}{26} \times \frac{64}{63} \times$ &c. = ,211466250444; hence $\frac{8}{7} \times \frac{27}{26} \times \frac{64}{63} \times$ &c. = 1,627295, &c.

Hence we may find the value of such a quantity, supposing the number of factors to be finite.

Ex. To find the value of $\frac{2}{1} \times \frac{4}{3} \times \frac{6}{5} \times$ &c. . . . . to $\frac{2x}{2x-1}$.

Here the general term being $\frac{2x}{2x-1}$, we have $\frac{\dot{\pi}}{\pi} = -\frac{\dot{x}}{2x^2 \cdot x} = -\frac{\dot{x}}{2x^2} - \frac{\dot{x}}{4x^3} - \frac{\dot{x}}{8x^4} - \frac{\dot{x}}{16x^5}$ − &c.; hence hyp. log. $\frac{2x}{2x-1} = \frac{1}{1 \cdot 2x} + \frac{1}{2 \cdot 4x^2} + \frac{1}{3 \cdot 8x^3} + \frac{1}{4 \cdot 16x^4}$ + &c.. Now write 2, 3, 4, &c. for x, and we have the hyp. log. $\frac{4}{3} \times \frac{6}{5} \times \frac{8}{7} \times$ &c. . . .

$$\frac{2x}{2x-1} = \left.\begin{array}{l} \frac{1}{2} \times \overline{\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots\dots \frac{1}{x}} \\ + \frac{1}{2 \cdot 4} \times \overline{\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots\dots \frac{1}{x^2}} \\ + \frac{1}{3 \cdot 8} \times \overline{\frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \dots\dots \frac{1}{x^3}} \\ + \quad \text{\&c.} \qquad \text{\&c.} \qquad \text{\&c.} \end{array}\right\}$$

But, by Prop. 15. $\frac{1}{2} \times \overline{\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots\dots \frac{1}{x}} = \frac{1}{2}$hyp. log. $x - ,211392167549 + \frac{1}{4n} + \frac{5}{24n^2} + \frac{1}{6n^3} + \frac{31}{240n^4} + \frac{1}{10n^5} + \frac{41}{504n^6}$ + &c.; also $\frac{1}{2 \cdot 4} \times \overline{\frac{1}{2^2} + \frac{1}{3^2} + \dots\dots \frac{1}{x^2}} = \frac{1}{2 \cdot 4} \times \overline{A - \frac{1}{n} - \frac{1}{2n^2} - \frac{1}{6n^3} + \frac{1}{30n^5} - \text{\&c.}}$ and $\frac{1}{3 \cdot 8} \times \overline{\frac{1}{2^3} + \frac{1}{3^3} + \dots\dots \frac{1}{x^3}} = \frac{1}{3 \cdot 8} \times \overline{B - \frac{1}{2n^2} - \frac{1}{2n^3} - \frac{1}{4n^4} + \frac{1}{12n^6} - \text{\&c.}}$, and so on for the other serieses: hence, by substitu-

tion, hyp. log. $\frac{4}{3} \times \frac{6}{5} \times \frac{8}{7} \times$ &c. . . . . $\frac{2x}{2x-1} = \frac{1}{2}$ hyp. log. $x -$ $,12078223764 + \frac{1}{8n} + \frac{1}{8n^2} + \frac{23}{192n^3} + \frac{7}{64n^4} + \frac{61}{640n^5}$ + &c.; consequently $\frac{4}{3} \times \frac{6}{5} \times \frac{8}{7} \times$ &c. . . . . $\frac{2x}{2x-1}$ = the natural number corresponding to the right hand side of the equation; hence $\frac{2}{1} \times \frac{4}{3} \times \frac{6}{5} \times$ . . . . . $\frac{2x}{2x-1}$ = twice that natural number.

Ex. Let $x = 10000$; then

$$\begin{aligned} \tfrac{1}{2} \text{ hyp. log. } x &= 4,605170185988 \\ \text{const. quant. } &= ,120782237640 \\ \hline & 4,484387948348 \\ \frac{1}{8n} &= ,000012498750 \\ \frac{1}{8n^2} &= ,000000001249 \\ \hline & 4,484400448347 \end{aligned}$$

the natural number corresponding to which hyp. log. is 88,6238, &c., consequently $\frac{2}{1} \times \frac{4}{3} \times \frac{6}{5} \times$ . . . . . . $\frac{20000}{19999} = 177,2476$.

If x be a very large number, it may be sufficiently exact in most cases to take twice the natural number corresponding to the hyp. log. of $\frac{1}{2}$ hyp. log. $x - ,120782237640$.